Nonlocal incoherent white-light solitons in logarithmically nonlinear media

Ming Shen,* Qi Wang, Jielong Shi, Yuanyuan Chen, and Xinglin Wang
Department of Physics, Shanghai University, 99 Shangda Road, Shanghai 200444, People's Republic of China
(Received 11 March 2005; revised manuscript received 2 June 2005; published 10 August 2005)

The propagation properties of white-light solitons in spatially nonlocal media with a logarithmically nonlinearity are investigated theoretically. The existence curve of the stationary nonlocal incoherent soliton is obtained and the coherence characteristics of the soliton are also described. The evolution behaviors of the nonlocal white-light soliton are discussed in detail by both approximate analytical solution and numerical simulation when the solitons undergo periodic oscillation.

DOI: 10.1103/PhysRevE.72.026604 PACS number(s): 42.65.Tg, 42.65.Jx

I. INTRODUCTION

The propagation of incoherent solitons in noninstantaneous nonlinear media has drawn considerable attention in the past decade [1–16]. Experiments on incoherent solitons [1,2] have initiated a great deal of theoretical research on understanding solitons made from incoherent light. In recent years, fully incoherent white-light solitons were investigated theoretically by both numerical methods [12] and the mutual spectral density method [11,13,14]. Last year, spontaneous pattern formation of white-light solitons was also studied experimentally [15].

Most recently, the propagation of optical beams in nematic liquid crystals has attracted great interest. A liquid crystal has a strong noninstantaneous nonlinearity response which is sufficiently slow to allow for formation of both coherent and incoherent solitons [17–19]. Another interesting property of liquid crystal is inherent *spatially nonlocality* [20], which is a key feature associated with light-induced molecular orientation [19,21].

Spatially nonlocal nonlinearity is a universal phenomenon of many physical systems especially in the propagation of nonlinear optical beams and soliton formation. After the pioneering work on self-focusing and filamentation in capillaries by Braun [22], experiments on optical solitons in liquid crystal have been reported extensively, e.g., formation of multiple solitons [23], accessible solitons [24], the modulation instability of solitons [25–27], and discrete propagation of solitons in nematic liquid crystal [28]. Solitons in nonlocal nonlinear medium have been also discussed theoretically, e.g., the collapse arrest of finite-size beams [29], attraction and formation of dark solitons [30], modulation instability in defocusing materials [31,32], large phase shift of nonlocal solitons [33], quadratic solitons [34], nonlocal nonlinear photonic lattices [35,36] and vortex solitons in nonlocal nonlinear media [37,38].

Recently Peccanti *et al.* reported the first observation of incoherent solitons [17] and their interaction [18,19] in nematic liquid crystals. Krolikowski *et al.* analyzed the effect of nonlocality on the propagation of partially coherent beams and formation of incoherent solitons theoretically [39]. In this paper we discuss the propagation properties of temporally and spatially incoherent solitons (*white-light solitons*) in nonlocal nonlinear media. Here we assume the nonlinearity is of logarithmical type [40] because incoherent beams propagate in both local [6,7,13] and nonlocal media [39] and fully coherent beams propagate in nonlocal media [41] with a logarithmically nonlinearity and have an exact analytical stationary solution. We use mutual spectrum density theory [11,13,14] to discuss the properties of simultaneous *spatiotemporal incoherence and nonlocality* (to our knowledge, this is the first time). We obtain the existence curve and the analytical expression of such nonlocal incoherent solitons. This incoherent soliton has an elliptic Gaussian intensity profile and elliptic Gaussian spatial correlation statistics. The initial coherence characteristics of the beam and the properties of the medium (*nonlinearity and nonlocality*) decide the propagation of the beam. When the soliton undergoes periodic oscillation, we discuss the evolution behaviors of the nonlocal white-light soliton by both approximate analytical solution and numerical simulation.

II. WHITE-LIGHT SOLITONS IN LOGARITHMICALLY NONLINEAR NONLOCAL MEDIA

The effective theory for treating the propagation of spatially and temporally incoherent light is the so-called mutual spectral density theory [11,13,14]. Consider a spatially and temporally incoherent light beam (the light source used in Ref. [2]) with the temporal power spectrum $[\omega_{min},\omega_{max}]$. The propagation of the incoherent beam satisfies an integro-differential equation [11,13,14]

$$\frac{\partial B_\omega}{\partial z}-\frac{i}{2k_\omega}[\Delta_{\perp 1}^2-\Delta_{\perp 2}^2]B_\omega = \frac{ik_\omega}{n_0}\{\delta n(I(\mathbf{r}_1,z))-\delta n(I(\mathbf{r}_2,z))\}B_\omega(\mathbf{r}_1,\mathbf{r}_2,z), \quad (1)$$

where $k_\omega=n_0\omega/c$ is the wave vector and $B_\omega(\mathbf{r}_1,\mathbf{r}_2,z)$ represents the mutual spectral density for all the frequency constituents, which describes the correlation statistics between the electric field values at two different spatial points upon the transverse section of the beam [13]; the response of the material is $n^2(I)=n_0^2+2n_0\delta n(I)$, where n_0 and $\delta n(I)$ denote

*Electronic address: shenmingluck@graduate.shu.edu.cn

the linear and nonlinear parts of the refractive index. In nonlocal nonlinear media we take the logarithmical nonlinear refractive index change δn to be the following nonlocal function of light intensity [39]:

$$\delta n(\vec{r},I) = n_2 \ln\left[\int F(\vec{r}-\vec{\xi})I(\vec{\xi})d\vec{\xi}\right] \tag{2}$$

where

$$I(\mathbf{r},z) = 1/2\pi\int_0^\infty d\omega\, B_\omega(\mathbf{r},\mathbf{r},z) \tag{3}$$

denotes the time-averaged intensity; $\int d\vec{r}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dx\,dy, n_2$ specifies the strength of the nonlinearity, and $F(\vec{r})=F(r)$ is the nonlocal response function, which depends only on the spatial length. From Eq. (2) it is easy to show that the refractive index changed in a particular point in space is determined not only by the light intensity in this point, but also by the intensity in a certain surrounding beyond this point associated with the width of the response function. The width of the nonlocal response function determines the degree of nonlocality [42]. In the limit of a singular response, $F(\vec{r})=\delta(r)$, the nonlinearity is a local function of the intensity $\delta n = n_2\ln(I)$. In the limit of a strong nonlocality, i.e., when the width of the response function is much broader than the intensity profile of the beam, the medium should be treated as a linear medium [41,43].

In order to make an analytical description in a logarithmic medium we assume that the nonlocality is described by the normalized Gaussian response function [39]:

$$F(r) = \frac{1}{\pi\sigma^2}\exp\left(-\frac{x^2+y^2}{\sigma^2}\right), \tag{4}$$

where σ is the degree of the nonlocality [34]. Also we assume the nonlocal medium is isotropic.

For convenience we introduce new spatial coordinates $\vec{r}$ and $\vec{\rho}$,

$$\vec{r} = \frac{\vec{r}_1+\vec{r}_2}{2}, \quad \vec{\rho} = \vec{r}_1-\vec{r}_2, \tag{5}$$

where the spatial vector $\vec{r}$ and the vector $\vec{\rho}$ are the midpoint and difference coordinates of the mutual spectrum density in the new system [13,44].

Combining the above equations we obtain the nonlinear propagation equation of fully incoherent beam in nonlocal media,

$$\frac{\partial B_\omega}{\partial z} - \frac{i}{k_\omega}\left(\frac{\partial^2}{\partial r_x\,\partial \rho_x}+\frac{\partial^2}{\partial r_y\,\partial \rho_y}\right)B_\omega = \frac{ik_\omega n_2}{n_0}\ln\left(\frac{\int F(\vec{r}+\vec{\rho}/2-\vec{\xi})I(\vec{\xi})d\vec{\xi}}{\int F(\vec{r}-\vec{\rho}/2-\vec{\xi})I(\vec{\xi})d\vec{\xi}}\right)B_\omega(\vec{r},\vec{\rho},z). \tag{6}$$

Since incoherent solitons propagating in logarithmical nonlinear medium maintain elliptic Gaussian intensity profiles and elliptic Gaussian correlation for both local [5] and nonlocal media [39], we seek for the mutual spectral density of the stationary Gaussian-Schell beam in the form [13]

$$B_\omega(\vec{r},\vec{\rho},z=0) = A_\omega(0)\prod_{j=x,y}\exp-\left(\frac{r_j^2}{2R_j^2(0)}+\frac{\rho_j^2}{2Q_j^2(0)}\right), \quad j=x,y. \tag{7}$$

Here $A_\omega(0)$ denotes the spectral density of the light beam; $R_x(0)$ and $R_y(0)$ denote the characteristic width of the spatial soliton; $Q_x(0)$ and $Q_y(0)$ denote the initial effective coherence radius for different frequency constituents which are given by the relation [7,39]

$$\frac{1}{Q_j^2(0)} = \frac{1}{r_{cj}^2(0)} + \frac{1}{4R_j^2(0)}, \quad j=x,y, \tag{8}$$

where $r_{cj}(0)$ is the initial coherence radius for every frequency and $R_j(0)$ is the initial width of the beam. Due to the logarithmical form of the nonlinearity, the beam will maintain Gaussian statistics during the propagation process we look for the general solution to Eq. (6) in the following form:

$$B_\omega(r_x,\rho_x,r_y,\rho_y,z) = A_\omega(z)\prod_{j=x,y}\exp\left(-\frac{r_j^2}{2R_j^2(z)}-\frac{\rho_j^2}{2Q_j^2(z)} + ir_j\rho_j\phi_\omega(z)\right) \tag{9}$$

where $A_\omega(z)$ and $\phi_\omega(z)$ denote the amplitude and the phase of the mutual spectral density, respectively; $R_j(z)$ and $Q_j(z)$ are the width and effective coherence radius of the beam, respectively. The initial conditions are $A_\omega(0), R_j(0), Q_j(0)$, and $\phi_\omega(z=0)=0$. Inserting Eq. (9) into Eq. (6), we obtain a set of ordinary differential equations for the parameters of the mutual spectral density,

$$\frac{dQ_j(z)}{dz} = \frac{1}{k_\omega}\phi_\omega(z)Q_j(z), \tag{10}$$

$$\frac{dR_j(z)}{dz} = \frac{1}{k_\omega}\phi_\omega(z)R_j(z), \tag{11}$$

$$\frac{dA_\omega(z)}{dz} = -\frac{2}{k_\omega}\phi_\omega(z)A_\omega(z), \tag{12}$$

$$\frac{1}{k_\omega}\frac{d\phi_\omega(z)}{dz} = \frac{1}{k_\omega^2}\frac{1}{Q_j^2(z)R_j^2(z)} - \frac{\phi_\omega^2(z)}{k_\omega^2} - \frac{n_2}{n_0}\frac{1}{R_j^2(z)+\sigma^2}. \tag{13}$$

From the first two equations (10) and (11) we obtain the relation

$$Q_j(z)/R_j(z) = Q_j(0)/R_j(0) \tag{14}$$

which shows that during the evolution the beam conserves its coherence. The wider (narrower) the beam, the larger (smaller) the coherence radius. Combining Eqs. (11) and (12) gives the amplitude $A_\omega(z)=R_j^2(0)/R_j^2(z)A_\omega(0)$. It shows that the amplitude of stationary solitons is not dependent on whether or not the nonlinearity is local. Finally, inserting Eq. (13) into Eq. (11),we obtain the evolution equation of the incoherent beam,

$$\frac{d^2R_j(z)}{dz^2} - \frac{1}{k_\omega^2}\frac{R_j^2(0)}{R_j^3(z)Q_j^2(0)} + \frac{n_2}{n_0}\frac{R_j(z)}{R_j^2(z)+\sigma^2} = 0. \tag{15}$$

This equation describes the dynamics of the spatially and temporally incoherent beamwidth in a nonlocal media with a logarithmical nonlinearity. The dynamic is a process that self-focusing induced by the nonlinearity compensate the spreading induced by the incoherent diffraction and the nonlocality. If $n_2=0$ or $\sigma\to\infty$, Eq. (15) describes the simple diffraction of the white-light beam. When we take $\sigma=0$, the equation describes a white-light soliton in a local medium [13]. If $n_2<0$, this is a self-defocusing nonlinearity which will enhance the natural spreading of the beam. Here we will only concentrate on the focusing case with $n_2>0$.

Nonlocal white-light soliton solutions are obtained from Eq. (15) by setting $R_j(z)=R_j(0)$, which gives the relation

$$n_2 = \frac{n_0}{k_\omega^2}\frac{1}{Q_j^2(0)}\left(1+\frac{\sigma^2}{R_j^2(0)}\right). \tag{16}$$

This relation shows that in order to get a stationary white-light soliton, the initial effective coherence radius $Q_j(0)$ should obey

$$Q_j(0) = \frac{1}{k_\omega}\sqrt{\frac{n_0}{n_2}\left(1+\frac{\sigma^2}{R_j^2(0)}\right)} = \tilde{Q}_j(0)\sqrt{1+\frac{\sigma^2}{R_j^2(0)}}, \tag{17}$$

$$\tilde{Q}_j(0) = Q_j(0)(\sigma=0) = \frac{c}{\omega\sqrt{n_0n_2}} = \frac{\omega_0}{\omega}Q_0, \tag{18}$$

where $Q_0=c\omega_0^{-1}/\sqrt{n_0n_2}$ and ω_0 denotes the central frequency within the power spectrum. Quantities $Q_j(0)$ are determined by the strength of the nonlinearity n_2, the frequency ω, the linear index of refraction n_0, the speed of light c, the width of the beam $R_j(0)$, and the degree of the nonlocality σ. From Eqs. (17) and (18) we also know that trapping of a white-light beam with a given nonlinearity n_2 in a nonlocal medium requires a larger effective initial coherence radius than that in the case of a local nonlinear response. This is due to the fact that the nonlocality effectively leads to a decrease of the strength of focusing and subsequently weakens the localization of the beam [39].

From Eqs. (8) and (17), it follows that the characteristic widths of the white-light beam are connected with the spatial correlation distances $r_{cj}(0)$, the strength of the nonlinearity n_2, and the degree of the nonlocality σ through

$$\frac{1}{r_{cj}(0)} = \sqrt{\frac{\omega^2n_0n_2R_j^2(0)}{c^2[R_j^2(0)+\sigma^2]} - \frac{1}{4R_j^2(0)}}. \tag{19}$$

Equation (19) is the existence curve for the white-light solitons in a nonlocal medium with a logarithmical nonlinear response. In the special case when $\sigma=0$, our result correctly reduces to the solution of white-light solitons in a local logarithmical nonlinear medium previously obtained by Buljan [13].

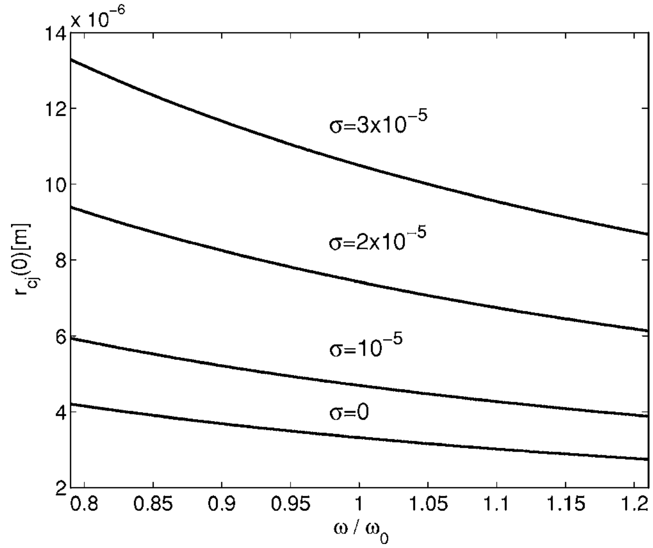

FIG. 1. The spatial correlation distance deceases with the increase of frequency for different degrees of the nonlocality. The value of $r_{cj}(0)$ is calculated from Eq. (19) with the following parameters: $n_2=0.0003$, $n_0=2.3$, $R_j(0)=10\ \mu$m, and $\omega_0=3.44\times10^{15}$ Hz, which corresponds to the wavelength of 547 nm in vacuum.

Figure 1 shows the relation between the spatial correlation distance and frequency as calculated from Eq. (19) for realistic parameter values. From the existence curve we find that for stationary white-light solitons to exist in nonlocal medium, the spatial correlation distance should larger for lower frequencies and smaller for higher frequencies at a given degree of the nonlocality σ; the spatial correlation distance should also increase with the increase of the degree of the nonlocality.

This result can be explained as follows. A nonlocal spatially incoherent solitons occurs when incoherent diffraction and the decrease of the strength of focusing induced by the nonlocality are exactly balanced by the nonlinearity. White-light solitons are made up of many wavelengths; they are all trapped within the same waveguide and have approximately the same incoherent diffraction angle, $\theta\propto\lambda/r_{cj}$ (λ is the wavelength), so we can obtain $r_{cj}\propto\lambda$; i.e., the spatial correlation distance of a white-light soliton is larger at lower frequencies and shorter at higher frequencies [12]. In addition, the nonlocality will weaken the localization of the beam, so the initial coherence properties of the incident beam should be larger than that in the local medium.

From Eq. (19), it illustrates that for a nonlocal white-light soliton to exist, the characteristic width must be larger than a threshold value,

$$R_j(0) > \sqrt{\frac{c^2+\sqrt{c^4+16n_0n_2c^2\sigma^2\omega^2}}{8n_0n_2\omega^2}}. \tag{20}$$

Equation (20) should be satisfied for every frequency within the spectrum. Consider the spectrum $[\omega_{min},\omega_{max}]$, where

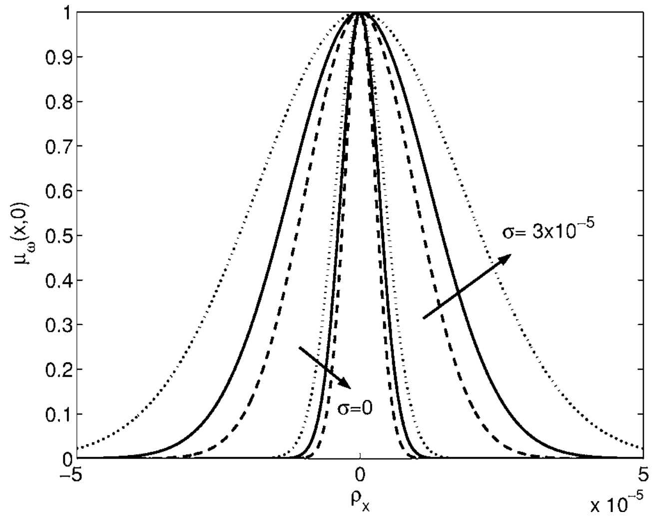

FIG. 2. Complex coherence factor $\mu_\omega(x,0)$ of a stationary nonlocal white-light soliton for different degrees of nonlocality at three representative frequencies: $\omega_{min}=2.69\times10^{15}$ Hz (dotted line), $\omega_0=3.44\times10^{15}$ Hz (solid line), $\omega_{max}=4.19\times10^{15}$ Hz (dashed line).

$\omega_{min}=\omega_0(1-\epsilon)$ and $\omega_{max}=\omega_0(1+\epsilon)$, where ϵ denotes the width of the spectrum, so if

$$R_j(0) > \sqrt{\frac{c^2+\sqrt{c^4+16n_0n_2c^2\sigma^2\omega_0^2(1-\epsilon)^2}}{8n_0n_2\omega_0^2(1-\epsilon)^2}} \qquad (21)$$

Eq. (20) is satisfied for every frequency. This means that for a nonlocal soliton to exist, the width of the beam should exceed a value determined by the width of the temporal spectrum ϵ, the strength of the nonlinearity n_2, and the degree of the nonlocality σ.

The spatial coherence properties of the soliton are described in terms of the complex coherence factor at frequency ω [45]:

$$\mu_\omega(\mathbf{r}_1,\mathbf{r}_2,z)=\frac{B_\omega(\mathbf{r}_1,\mathbf{r}_2,z)}{\sqrt{B_\omega(\mathbf{r}_1,\mathbf{r}_1,z)B_\omega(\mathbf{r}_2,\mathbf{r}_2,z)}}. \qquad (22)$$

The quantity $\mu_\omega(\mathbf{r}_1,\mathbf{r}_2,z)$ is referred to as the spectral degree of coherence at frequency ω, or the complex degree of spatial coherence at frequency ω.

Combining Eqs. (7), (17), and (22), we can obtain the following relation

$$\mu_\omega(r_x,\rho_x,r_y,\rho_y) = \prod_{j=x,y}\exp\left[\left(\frac{1}{8R_j^2(0)}-\frac{n_0n_2\omega^2}{2c^2}\frac{R_j^2(0)}{R_j^2(0)+\sigma^2}\right)\rho_j^2\right]. \qquad (23)$$

It is easy to see that the complex coherence function $\mu_\omega(r_x,\rho_x,r_y,\rho_y)$ depends only on the difference coordinates ρ_j at a given σ. This means that the fluctuations of the light that forms the soliton obey a stationary random process [45]. Figure 2 shows the complex coherence factor $\mu_\omega(x,0)$ at three representative frequencies ω_{min},ω_0, and ω_{max} for two different degrees of nonlocality $\sigma=0$ and 3×10^{-5}.

We find that the spatial correlation distance is larger for lower frequencies and smaller for higher frequencies, as can be seen from the width of $\mu_\omega(x,0)$; it is as same as the conditions for white-light solitons to exist in a local medium [12]. We also find that the spatial correlation distance increases with the nonlocality. This is because the nonlocality decrease the strength of beam focusing; the nonlocality is an inherent property of the medium and it is difficult to control. So to obtain a white-light soliton the coherence properties of the incident beams must be enhanced over those in the local medium.

The intensity profile of the stationary white-light soliton can be obtained through Eq. (3)

$$I(x,y,z)=I_0\exp\left(-\frac{x^2}{2R_x^2(0)}-\frac{y^2}{2R_y^2(0)}\right), \qquad (24)$$

where I_0 is the peak intensity of the beam. We can see that the spatial intensity profile maintains the Gaussian elliptic profile and the intensity is independent of the nonlocality for a stationary white-light soliton.

In this section, we discuss the temporally and spatially incoherent solitons (white-light solitons) in a nonlocal medium with a logarithmic nonlinearity. We find an analytic solution representing such incoherent solitons. This incoherent soliton has an elliptic Gaussian intensity profile and elliptic Gaussian spatial correlation statistics. The existence curve of the soliton connects the strength of the nonlinearity, the spatial correlation distance, the characteristic width of the soliton, and the degree of the nonlocality. For white-light solitons to exist, the spatial correlation distance should be larger at lower frequencies and shorter at higher frequencies. The initial coherence properties should be larger in a nonlocal medium than in a local medium.

III. STABILITY AND PERIODIC OSCILLATION OF NONLOCAL WHITE-LIGHT SOLITON

In this section, we discuss the evolution properties of the nonlocal white-light solitons that are both stable and undergo periodic oscillation. In the above section, we have obtained the existence curve for stationary nonlocal white-light solitons to exist [see Eq. (7) and (19)] and we believe the soliton is stable. Here we will use the approaches from Refs. [7] and [13] for incoherent solitons in a local logarithmic medium to discuss the stability and periodic oscillation of the nonlocal white-light soliton. We assume that the mutual spectral density is slightly different from the Eq. (9) [13]

$$B_\omega(r_x,\rho_x,r_y,\rho_y,z)=A_\omega(z)\prod_{j=x,y}\exp\left(-\frac{r_j^2}{2R_j^2(z)}-\frac{\rho_j^2}{2q_j^2(z)}\frac{\omega^2}{\omega_0^2} + ir_j\rho_j\phi_\omega(z)\right). \qquad (25)$$

Similarly $A_\omega(z)$ and $\phi_\omega(z)$ denote the amplitude and the phase of the mutual spectral density, respectively; $R_j(z)$ is the beam width and $q_j(z)$ is the effective coherence radius for frequency ω_0 with the following relation: $1/q_j^2(z)=1/r_{aj}^2(z)+1/4R_j^2(z)$ (here $r_{aj}(z)$ is the coherence radius for frequency

ω_0 during the propagation process). Inserting the expression (25) into Eq. (6), we obtain a set of ordinary differential equations for the parameters of the mutual spectral density, which is as same as Eqs. (10)–(12) except that Eq. (13) should change into

$$\frac{1}{k_\omega}\frac{d\phi_\omega(z)}{dz} = \frac{1}{k_{\omega_0}^2}\frac{1}{q_j^2(z)R_j^2(z)} - \frac{\phi_\omega^2(z)}{k_\omega^2} - \frac{n_2}{n_0}\frac{1}{R_j^2(z)+\sigma^2}. \tag{26}$$

If we let $q_j(z)=(c/\omega_0\sqrt{n_0 n_2})\sqrt{1+\sigma^2/R_j^2(0)}$ and $\phi_\omega(z)=0$, Eq. (26) gives the existence of a white-light soliton in a nonlocal medium [see Eq. (17)]. Combining Eq. (11) and (26), we obtain the evolution equation of the incoherent beam,

$$\frac{d^2R_j(z)}{dz^2} - \frac{1}{k_{\omega_0}^2}\frac{R_j^2(0)}{R_j^3(z)q_j^2(0)} + \frac{n_2}{n_0}\frac{R_j(z)}{R_j^2(z)+\sigma^2} = 0. \tag{27}$$

We can use the previous method [13] to analyze the propagation properties of the nonlocal white-light soliton. Here we utilize the form of Newton's equation and Jacobian elliptical function [7] to discuss the evolution properties of the beam. Assuming that $[dR_j(z)/dz]_{z=0}=0$, and integrating Eq. (27) once, we can obtain Newton's equation

$$\left(\frac{dR_j(z)}{dz}\right)^2 + P(R_j(z)) = 0 \tag{28}$$

for an effective particle moving in the potential $P[R_j(z)]$, which is given by

$$P(R_j(z)) = \frac{1}{k_{\omega_0}^2 q_j^2(0)}\left(\frac{R_j^2(0)}{R_j^2(z)} - 1\right) + \frac{n_2}{n_0}\ln\left(\frac{R_j^2(z)+\sigma^2}{R_j^2(0)+\sigma^2}\right). \tag{29}$$

The asymmetric potential of an incoherent white-light beam is illustrated in Fig. 3 for different degrees of nonlocality σ. As nonlocality increases the width of the potential also increases while its minimum decreases. This result may be close to the potential of a partially coherent beam in a self-focusing medium [39].

A stationary nonlocal white-light soliton (corresponding to the effective particle being located at the bottom of the potential well) can be obtained by $[\partial P(R_j(z))/\partial R_j(z)]_{R_j(z)=R_j(0)}=0$,

$$\Delta = \frac{n_2}{n_0}\frac{2R_j^2(0)}{R_j^2(0)+\sigma^2} - \frac{2}{k_{\omega_0}^2 q_j^2(0)} = 0 \tag{30}$$

where Δ is the detuning [7] and denotes the relation between the initial coherence properties of incident beam and the properties of the medium (nonlinearity, nonlocality). Combining Eqs. (8) and (30), we find the expression for the radius of the stationary nonlocal white light soliton

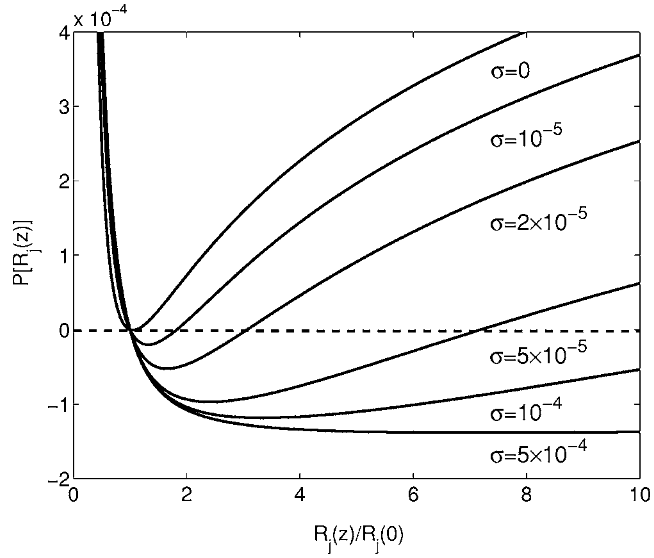

FIG. 3. Potential $P(R_j(z))$ in a focusing nonlinear medium for different degrees of nonlocality. The initial parameters are $n_2=0.0003$, $n_0=2.3$, $R_j(0)=10$ μm, $\omega_0=3.44\times10^{15}$ Hz, and $q_j(0)=3.16$ μm.

$$R_j^2(0) = \frac{\tilde{R}_j^2(0)}{2}\left(1+\frac{4\sigma^2}{r_{aj}^2(0)} + \sqrt{\left(1+\frac{4\sigma^2}{r_{aj}^2(0)}\right)^2 + \frac{4\sigma^2}{\tilde{R}_j^2(0)}}\right) \tag{31}$$

where $\tilde{R}_j^2(0)$ is the expression for the radius of the white-light soliton in a local medium which can be obtained from Eq. (30) for $\sigma=0$:

$$\tilde{R}_j^2(0) = R_j^2(0)(\sigma=0) = \frac{1}{4[n_2k_{\omega_0}^2/n_0 - 1/r_{aj}^2(0)]}. \tag{32}$$

In Fig. 4 we plot the normalized radius of nonlocal white-light soliton versus the degree of nonlocality σ for different

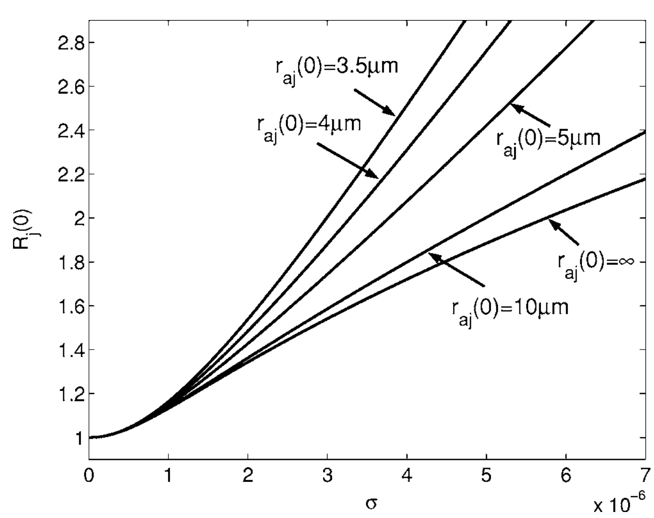

FIG. 4. Normalized soliton width as a function of the degree of nonlocality for different coherence radius. The initial parameters are $n_2=0.0003$, $n_0=2.3$, and $\omega_0=3.44\times10^{15}$ Hz.

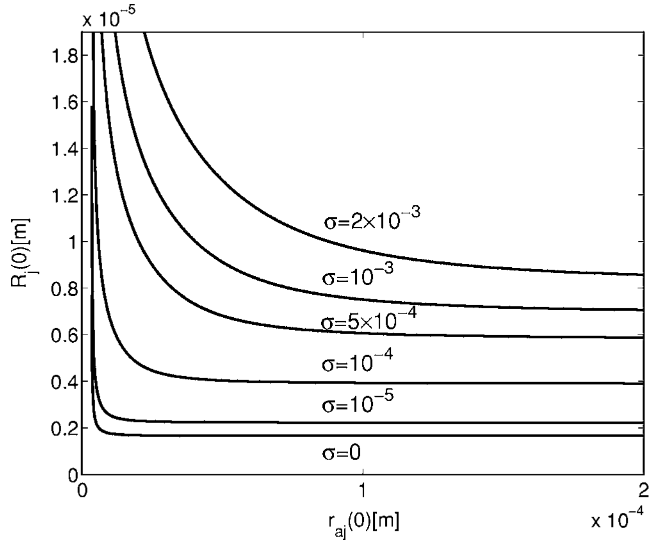

FIG. 5. Soliton width as a function of the coherence radius for different degree of the nonlocality. The initial parameters are $n_2 = 0.0003$, $n_0=2.3$, and $\omega_0=3.44\times10^{15}$ Hz.

coherence radius $r_{aj}(0)$. It is clearly shows that the soliton radius increase with nonlocality and becomes proportional to σ in highly nonlocal regime. We can see that when $\sigma/r_{aj}(0)\gg1$

$$R_j(0)=2\tilde{R}_j(0)\frac{\sigma}{r_{aj}(0)}. \tag{33}$$

Because the degree of the nonlocality is an inherent property of the medium we can vary the coherence properties of an incident beam to control the soliton radius.

In Fig. 5 we plot the soliton radius versus the coherence radius $r_{aj}(0)$ for different nonlocality σ. It shows that the soliton radius increases as incoherence and nonlocality increase. Furthermore, at every given nonlocality σ, there exists the same threshold value for the coherence radius, below which the nonlocal white-light soliton cannot exist.

It is well known that for incoherent solitons in local media [6,7,46,47] and partially incoherent solitons in nonlocal media [39] there also exists a threshold. As to the nonlocal white-light soliton, the threshold value is determined by the strength of nonlinearily n_2, the linear part of the refractive index n_0, the frequency ω_0, and the coherence radius of the beam, through

$$\frac{n_2k_{\omega_0}^2}{n_0}-\frac{1}{r_{aj}^2(0)}>0. \tag{34}$$

We can see that this threshold value does not depend on the beam radius and the degree of the nonlocality. The reason has been illustrated by Krolikowski [39]. Physically, the formation of a nonlocal white-light soliton is a result of the interplay of nonlinearity, coherence, diffraction, and nonlocality, and the threshold value is the least condition for a white-light soliton to exist.

When initial conditions are different from the soliton solution, i.e., for nonzero detuning $\Delta\neq0$, the incoherent white-light beam will undergo periodic oscillations. To obtain the evolution behaviors of the unstable incoherent soliton, we can numerically integrate Eq. (28) directly. In a special condition in the limit of small detuning, we can find an approximate analytical solution for the oscillations [7]. Here we discuss the evolution behaviors of the nonlocal white-light soliton by both approximate analytical solution and numerical simulation. First, we concentrate on the approximate analytical solution. For $|\Delta|\ll1$, the beam radius $R_j(z)$ will remain close to the initial value $R_j(0)$, and we expand the potential $P(R_j(z))$ around $R_j(z)=R_j(0)$. To third order $P(R_j(z))\approx P_3(R_j(z))$, where $P_3(R_j(z))$ is given by

$$P_3(R_j(z))=\Delta\theta+\alpha_{1j}\theta^2+\alpha_{2j}\theta^3. \tag{35}$$

Here $\theta=R_j(z)/R_j(0)-1$ is the variation of the normalized radius for the beam, and

$$\alpha_{1j}=\frac{3}{k_{\omega_0}^2q_j^2(0)}+\frac{n_2}{n_0}\frac{[\sigma^2-R_j^2(0)]R_j^2(0)}{[\sigma^2+R_j^2(0)]^2}, \tag{36}$$

$$\alpha_{2j}=\frac{-4}{k_{\omega_0}^2q_j^2(0)}+\frac{n_2}{3n_0}\frac{2R_j^6(0)-6R_j^4(0)\sigma^2}{[\sigma^2+R_j^2(0)]^3}. \tag{37}$$

Inserting Eq. (35) into Eq. (28) and integrating, we can obtain the solution of the beam radius in terms of the Jacobian elliptical sn function [7]

$$R_j(z)=R_j(0)\left[1+\theta_{j-}\text{sn}^2\left(\frac{\sqrt{\theta_{j+}\alpha_{2j}}}{2R_j(0)}z,m_j\right)\right],\quad m_j=\left|\frac{\theta_{j-}}{\theta_{j+}}\right|. \tag{38}$$

It shows that the beamwidth will periodically oscillate during the propagation process. Here $\theta_{j\pm}$ are the solutions of the equation $\alpha_{2j}\theta^2+\alpha_{1j}\theta+\Delta=0$. In the limit $|\Delta\ll1|$, we obtain that

$$\theta_{j-}=-\frac{n_0\Delta}{n_2}\frac{[\sigma^2+R_j^2(0)]^2}{4R_j^2(0)\sigma^2+2R_j^4(0)}, \tag{39}$$

$$\theta_{j+}=\frac{3[2\sigma^2+R_j^2(0)][\sigma^2+R_j^2(0)]}{5R_j^4(0)+15R_j^2(0)\sigma^2+6\sigma^4}. \tag{40}$$

According to $|\text{sn}|\leq1$, the maximal value of the beam width variety is

$$R_{jmax}(z)-R_j(0)=R_j(0)\theta_{j-}=-\frac{n_0\Delta}{n_2}\frac{[\sigma^2+R_j^2(0)]^2}{4R_j(0)\sigma^2+2R_j^3(0)}. \tag{41}$$

So the beamwidth variation depends on the sign of Δ. For positive detuning, the beamwidth decrease initially; it will oscillate between $R_j(0)$ and a somewhat lower value. For negative detuning, the beamwidth increases initially; it will oscillate between $R_j(0)$ and a somewhat larger value [7]. In Figs. 6 and 7 we plot periodic evolution of the beamwidth versus the propagation distance for negative Δ and positive Δ with different σ by both the approximate analytical solu-

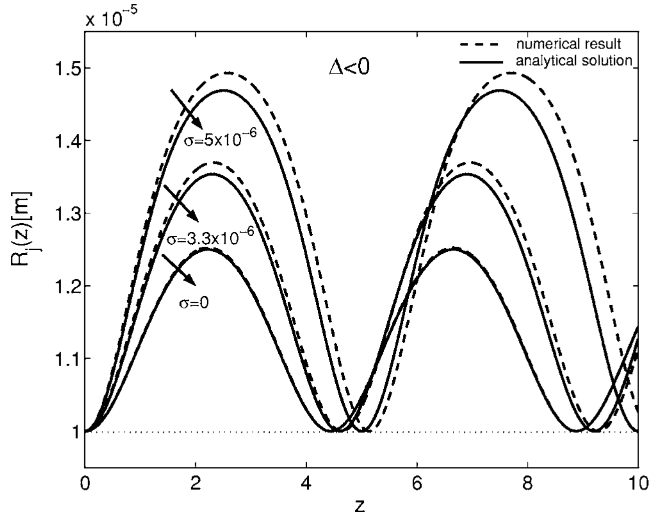

FIG. 6. Comparison of analytical and numerical results of soliton width oscillation for negative detuning at three representative degrees of the nonlocality: $\sigma=0$, 3.3×10^{-6}, and 5×10^{-6}. The solid lines represent analytical solutions and the dashed lines represent numerical results. The initial parameters are $n_2=0.0003$, $n_0=3$, $R_j(0)=10\ \mu\text{m}$, $\omega_0=3.44\times10^{15}$ Hz, and $q_j(0)=2.6\ \mu\text{m}$.

tion and numerical simulation. The approximate analytical solution is obtained from Eq. (38) with realistic parameters. For numerical simulation, we use the fourth-order Runge-Kutta method to integrate Eq. (28) directly. From these two figures we can see that the approximate analytical solution is in good agreement with the numerical result under the condition of small amplitude oscillation.

It is indeed the case that the approximate analytical solution is effective in studying the oscillation behaviors of the soliton as well as the numerical result. From Fig. 6 we can see that for negative detuning, the initial expansion effect for the beam induced by the incoherent and the nonlocality is stronger than the self-trapping effect induced by the nonlinearity; the beam will oscillate periodically with propagation distance between $R_j(0)$ and somewhat larger values. From Fig. 6 we can see, for a particular degree of the nonlocality, that the beamwidth will increase at first and decrease when it reaches the maximum. This is because the coherence properties of the beam will enhance with the increase of the beam width; when beamwidth reaches the maximum, the self-trapping effect will be stronger than the expansion effect, and the beamwidth will decrease. We also find that for negative detuning, the amplitude and the period of the beam oscillation will drastically increased for a high degree of nonlocality. The maximum of the beamwidth is also increased for high degree of nonlocality. This is because higher degrees of the nonlocality will enhance the absolute value of the detuning [see Eq. (30)], so the amplitude and period of the oscillation will increase synchronously so that the nonlocality will effectively lead to a decrease of the strength of focusing of the beam.

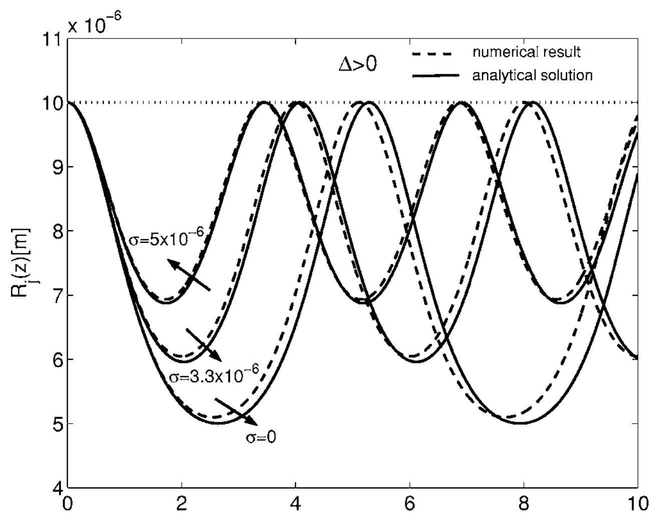

FIG. 7. Comparison of analytical and numerical results of soliton width oscillation for positive detuning at three representative degrees of the nonlocality: $\sigma=0, \sigma=3.3\times10^{-6}$ and 5×10^{-6}. The solid lines represent analytical solutions and the dashed lines represent numerical results. The initial parameters are $n_2=0.0006, n_0=3$, $R_j(0)=10\ \mu\text{m}, \omega_0=3.44\times10^{15}$ Hz, and $q_j(0)=2.9\ \mu\text{m}$.

Figure 7 illustrates the periodical evolution of the beam width for positive Δ with different σ. From Fig. 7 we can see that for positive detuning, the self-trapping effect induced by the nonlinearity is stronger than the initial expansion effect induced by the incoherent and the nonlocality, the beam will oscillate periodically with propagation distance between $R_j(0)$ and somewhat smaller values. From Fig. 7 we can see: for a particular degree of the nonlocality, the beamwidth will decrease at first and then increase after it reaches the minimum. This is because the coherence properties of the beam will decrease with the decrease of the beamwidth; when the beamwidth reaches the minimum, the self-trapping effect will be weaker than the expansion effect, and the beamwidth will increase again. But the oscillation behaviors of the positive detaning are quite different from those for negative detuning when the degree of the nonlocality increases. We find that for positive detuning, the amplitude and the period of the beam oscillation will drastically decrease for a high degree of nonlocality. But the minimum of the beamwidth increases for a higher degree of the nonlocality. This is because higher degrees of the nonlocality will reduce the value of the detuning for positive detuning [also see Eq. (30)], so the amplitude and period of the oscillation will decrease synchronously. In essence, the effect of the nonlocality is the same for both negative and positive detuning; the nonlocality leads to a decrease of the strength of focusing and subsequently to a weaker localization of the beam.

The coherence properties of the beam are determined by the complex coherence factor; combining Eqs. (22) and (25), we obtain the spatial coherence distance [45] along the x axis at a special propagation distance z,

$$d_x = \sqrt{1/\pi}\int_{-\infty}^{+\infty} |\mu_\omega(x_1,x_2,z)|^2 dx_2$$
$$= \frac{2q_x(0)R_x(0)}{\sqrt{4R_x^2(0)\omega^2/\omega_0^2 - q_x^2(0)}} \times\left[1 + \theta_{x-}\text{sn}^2\left(\frac{\sqrt{\theta_{x+}\alpha_{2x}}}{2R_x(0)}z, m_x\right)\right]. \quad (42)$$

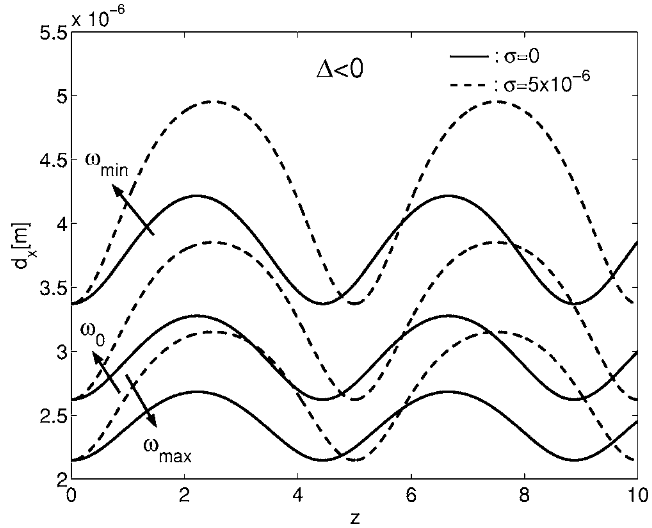

FIG. 8. Spatial coherence distance of a nonstationary nonlocal white-light soliton as a function of propagation distance for negative detuning at three representative frequencies $\omega_{min}=2.69\times10^{15}$, $\omega_0=3.44\times10^{15}$, $\omega_{max}=4.19\times10^{15}$ Hz. The solid lines represent $\sigma=0$ and the dashed lines represent $\sigma=5\times10^{-6}$. The initial parameters are the same as in Fig. 6.

As for the beamwidth, the coherence distance also oscillates with the propagation distance. Figures 8 and 9 illustrated the evolution of coherence radius at three representative frequencies for both negative and positive detuning. As in the above discussion, we only consider the approximate analytical solution here because it is in good agreement with the numerical result.

It is straightforward that the oscillation periods of the spatial coherence distance for every frequency are the same as the oscillation period of the beamwidth for a special degree

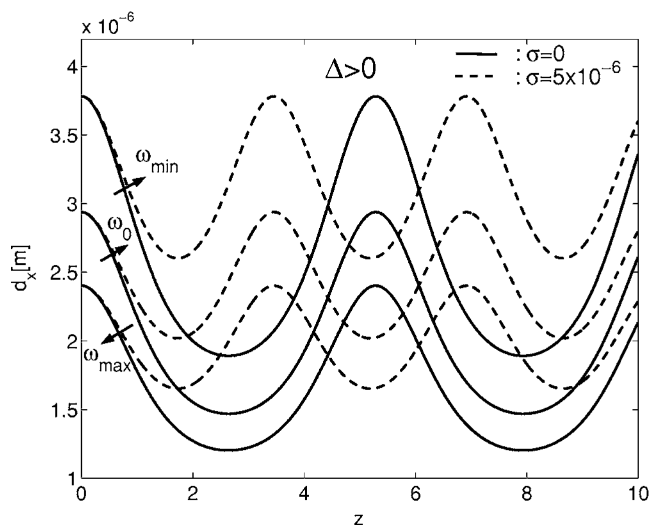

FIG. 9. Spatial coherence distance of a nonstationary nonlocal white-light soliton as a function of propagation distance for positive detuning at three representative frequencies $\omega_{min}=2.69\times10^{15}$, $\omega_0=3.44\times10^{15}$, $\omega_{max}=4.19\times10^{15}$ Hz. The solid lines represent $\sigma=0$ and the dashed lines represent $\sigma=5\times10^{-6}$. The initial parameters are the same as the Fig. 7.

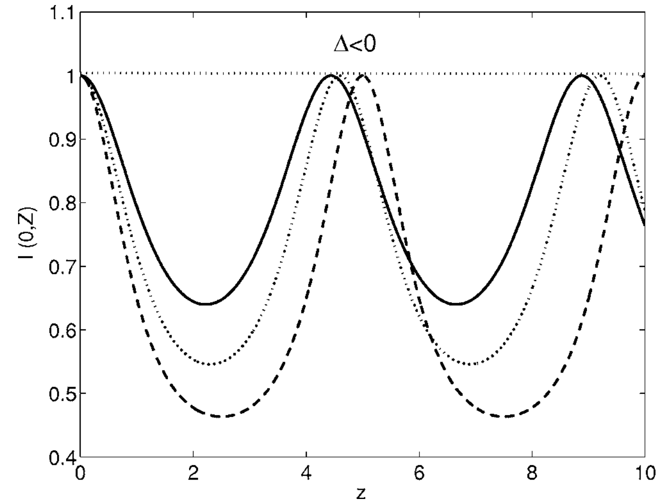

FIG. 10. The peak intensity of a nonstationary nonlocal white-light soliton as a function of propagation distance for negative detuning at three degrees of nonlocality $\sigma=0$ (solid line), 3.3×10^{-6} (dotted line), and 5×10^{-6} (dashed line). The initial parameters are the same as in Fig. 6.

of the nonlocality; the coherence distance will increase initially and then decrease for negative detuning while for positive detuning the coherence distance will decrease first and then decrease; the spatial coherence distance is larger for lower frequencies and smaller for higher frequencies for a specific degree of the nonlocality and independent of the sign of the detuning. But with the increase of the nonlocality, the amplitude and the period of the coherence distance for every frequency are enhanced for negative detuning and reduced for positive detuning, entirely like the oscillation behaviors of the beamwidth. It is true that when the beamwidth increases (decreases), the coherence distance increases (decreases), and the incohence properties will be weakened (strengthened).

When $z=0$ then $|\mathrm{sn}|=0$,

$$d_x=\frac{1}{\sqrt{\omega^2/q_x^2(0)\omega_0^2-1/4R_x^2(0)}}. \tag{43}$$

If $q_x(0)=(c/\omega_0\sqrt{n_0n_2})\sqrt{1+\sigma^2/R_x^2(0)}$, the expression is equal to Eq. (17), which represents a stationary nonlocal white-light soliton.

The intensity of the white-light beam (only considering the x direction) is obtained from Eq. (25) during the propagation process through

$$I(x,z)=\frac{I_0R_x^2(0)}{R_x^2(z)}\exp\left(-\frac{x^2}{2R_x^2(z)}\right) \tag{44}$$

where I_0 is the peak intensity of the incident beam. Figures 10 and 11 illustrate how the normalized peak intensity(at $x=0$) evolves with the propagation distance for negative and positive detuning, respectively.

These two figures show that the oscillation periods of the peak intensity of the beam are the same as the oscillation period of the beamwidth for a special degree of the nonlocality; the peak intensity of the beam will decrease initially

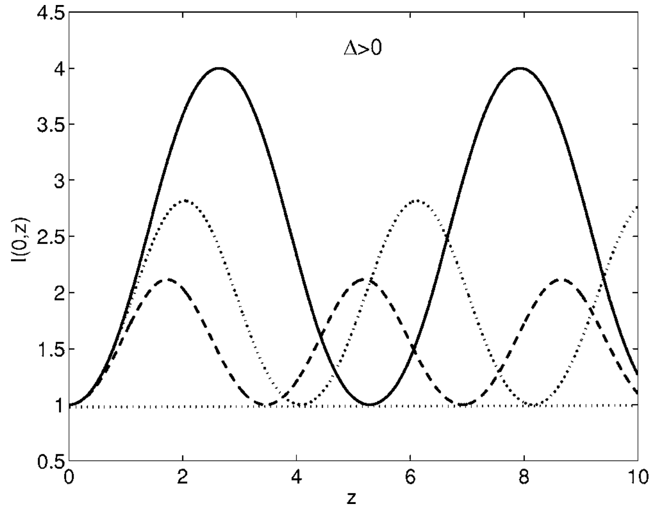

FIG. 11. The peak intensity of a nonstationary nonlocal white-light soliton as a function of propagation distance for positive detuning at three degrees of nonlocality $\sigma=0$ (solid line), 3.3×10^{-6} (dotted line), and 5×10^{-6} (dashed line). The initial parameters are the same as in Fig. 7.

and then increase for negative detuning while for positive detuning the peak intensity will increase initially and then decrease; with the increase of the nonlocality, the amplitude and the period of the peak intensity are enhanced for negative detuning and reduced for positive detuning, which is entirely similar to the oscillation behaviors of the beamwidth. This is because when the beamwidth increases (decreases), the density of the beam will decrease (increase) correspondingly; then the peak intensity of the beam will decrease (increase) synchronously.

In fact the intensity profile is slightly wider at lower frequencies and narrower with a higher peak at higher frequencies after some propagation distance for identical $I_\omega(x,z=0)$ of the input beam [12]. Here we only concentrate on the oscillation behaviors of the total peak intensity of the beam. In essence, the white-light soliton is a fully collective effect; all frequencies within the temporal spectrum participate in the formation of the soliton, and self-adjust their respective contributions [11,15]. Physically, all the temporal frequency constituents of the beam propagate in the same self-induced multimode waveguide and have different contributions in the formation of the soliton. The shorter wavelengths (higher frequencies) contribute more to the formation of the soliton than the longer wavelengths do [12,15].

IV. CONCLUSIONS

In summary, we have investigated the propagation properties of white-light solitons in spatially nonlocal nonlinear media with a logarithmic type of nonlinearity. We analytically demonstrate the evolution of white-light solitons in nonlocal media. This incoherent soliton has an elliptic Gaussian intensity profile and elliptic Gaussian spatial correlation statistics. The existence curve of the soliton connects the strength of the nonlinearity, the spatial correlation distance, the characteristic width of the soliton, and the degree of the nonlocality. When the soliton undergoes periodic oscillations, we discuss the evolution behaviors of the nonlocal white-light soliton by both an approximate analytical solution and a numerical simulation.

ACKNOWLEDGMENTS

We are grateful to Professor W. Krolikowski (The Australian National University, Australia) for his useful help and discussion. This work was supported in part by the Chinese National Natural Science Foundation (Grants No. 60177020, No. 60377025, and No. 60407007), the Science Foundation of Shanghai Municipal Commission of Education (Grant No. 01SG46), and the Shanghai Leading Academic Discipline Program.

[1] M. Mitchell, Z. Chen, M. Shih, and M. Segev, Phys. Rev. Lett. **77**, 490 (1996).
[2] M. Mitchell and M. Segev, Nature (London) **387**, 880 (1997).
[3] D. N. Christodoulides, T. H. Coskun, M. Mitchell, and M. Segev, Phys. Rev. Lett. **78**, 646 (1997).
[4] M. Mitchell, M. Segev, T. H. Coskun, and D. N. Christodoulides, Phys. Rev. Lett. **79**, 4990 (1997).
[5] D. N. Christodoulides, T. H. Coskun, M. Mitchell, and M. Segev, Phys. Rev. Lett. **80**, 2310 (1998).
[6] D. N. Christodoulides, T. H. Coskun, and R. I. Joseph, Opt. Lett. **22**, 1080 (1997).
[7] W. Krolikowski, D. Edmundson, and O. Bang, Phys. Rev. E **61**, 3122 (2000).
[8] Y. Y. Chen, Q. Wang, J. L. Shi, and Q. Wei, Acta Phys. Sin. **51**, 559 (2002).
[9] S. A. Ponomarenko, Phys. Rev. E **64**, 036618 (2001); **65**, 055601(R) (2002).
[10] M. I. Carvalho, T. H. Coskun, D. N. Christodoulides, M. Mitchell, and M. Segev, Phys. Rev. E **59**, 1193 (1999).
[11] H. Buljan, A. Siber, M. Soljacic, and M. Segev, Phys. Rev. E **66**, 035601(R) (2002).
[12] H. Buljan, M. Segev, M. Soljacic, N. K. Efremidis, and D. N. Christodoulides, Opt. Lett. **28**, 1239 (2003).
[13] H. Buljan, A. Siber, M. Soljacic, T. Schwartz, M. Segev, and D. N. Christodoulides, Phys. Rev. E **68**, 036607 (2003).
[14] H. Buljan, T. Schwartz, M. Segev, M. Soljacic, and D. N. Christodoulides, J. Opt. Soc. Am. B **21**, 397 (2004).
[15] T. Schwartz, T. Carmon, H. Buljan, and M. Segev, Phys. Rev. Lett. **93**, 223901 (2004).
[16] Yu. S. Kivshar and G. P. Agrawal, *Optical Solitons: From Fibers to Photonic Crystal* (Academic, San Diego, 2003).
[17] M. Peccianti and G. Assanto, Opt. Lett. **26**, 1791 (2001).
[18] M. Peccianti and G. Assanto, Phys. Rev. E **65**, 035603(R) (2002).
[19] M. Peccianti, K. A. Brzdakiewicz, and G. Assanto, Opt. Lett. **27**, 1460 (2002).

[20] M. Peccianti, C. Conti, and G. Assanto, Opt. Lett. **30**, 415 (2005).

[21] G. Assanto and IEEE J. Quantum Electron. **39**, 13 (2003).

[22] E. Braun *et al.*, Europhys. Lett. **23**, 239 (1993).

[23] X. Hutsebaut, C. Cambournac, M. Haelterman, A. Admski, and K. Neyts, Opt. Commun. **233**, 211 (2004).

[24] C. Conti, M. Peccianti, and G. Assanto, Phys. Rev. Lett. **91**, 073901 (2003); **92**, 113902 (2004).

[25] M. Peccianti, C. Conti, G. Assanto, A. D. Luca and C. Umeton, Nature (London) **432**, 733 (2004).

[26] M. Peccianti, C. Conti, and G. Assanto, Phys. Rev. E **68**, 025602(R) (2003).

[27] M. Kristian, K. Friedemann *et al.*, Opt. Lett. **29**, 280 (2004).

[28] A. Fratalocchi, G. Assanto, K. A. Brzdakiewicz, and M. A. Karpierz, Opt. Lett. **29**, 1530 (2004).

[29] O. Bang, W. Krolikowski, J. Wyller and J. J. Rasmussen, Phys. Rev. E **66**, 046619 (2002).

[30] N. I. Nikolov, D. Neshev, W. Krolikowski, O. Bang, J. J. Rasmussen, and P. L. Christiansen, Opt. Lett. **29**, 286 (2004).

[31] J. Wyller, W. Krolikowski, O. Bang, and J. J. Rasmussen, Phys. Rev. E **66**, 066615 (2002).

[32] W. Krolikowski, O. Bang, J. J. Rasmussen, and J. Wyller, Phys. Rev. E **64**, 016612 (2001).

[33] Q. Guo, B. Luo, F. Yi, S. Chi, and Yiqun Xie, Phys. Rev. E **69**, 016602 (2004).

[34] N. I. Nikolov, D. Neshev, O. Bang, and W. Z. Krolikowski, Phys. Rev. E **68**, 036614 (2003).

[35] Y. V. Kartashov, V. A. Vysloukh, and L. Torner, Phys. Rev. Lett. **93**, 153903 (2004).

[36] A. S. Desyatnikov *et al.*, Opt. Lett. **30**, 869 (2005).

[37] D. Briedis *et al.*, Opt. Express **13**, 435 (2005).

[38] A. I. Yakimenko, Y. A. Zaliznyak, and Yu. S. Kivshar (unpublished).

[39] W. Krolikowski, O. Bang, and J. Wyller, Phys. Rev. E **70**, 036617 (2004).

[40] Q. Wei, Q. Wang, and J. L. Shi, Acta Phys. Sin. **52**, 1645 (2003).

[41] A. Snyder and J. Mitchell, Science **276**, 1538 (1997); J. Opt. Soc. Am. B **16**, 236 (1999).

[42] W. Krolikowski and O. Bang, Phys. Rev. E **63**, 016610 (2000).

[43] W. Krolikowski, O. Bang, N. I. Nikolov, D. Neshev, J. Wyller, J. J. Rasmussen, and D. Edmundson, J. Opt. B: Quantum Semiclassical Opt. **6**, S288 (2004).

[44] S. M. Sears, M. Soljacic, D. N. Christodoulides, and M. Segev, Phys. Rev. E **65**, 036620 (2002).

[45] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, New York, 1995).

[46] A. Snyder and D. J. Mitchell, J. Opt. Soc. Am. B **13**, 1146 (1996).

[47] D. N. Christodoulides, T. H. Coskun, M. Mitchell, and M. Segev, Phys. Rev. Lett. **80**, 2310 (1998).